Centre Number

Student Number

SCEGGS Darlinghurst

2007

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 1

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\lim_{x \to 0} \dfrac{\sin 2x}{5x}$ **2**

(b) Solve for x: **2**

$$\frac{-2}{x-3} \leq 1$$

(c) Sketch the graph of $y = -3\sin^{-1}\dfrac{x}{2}$ clearly labeling all important features. **2**

(d) Evaluate $\displaystyle\int_0^{\frac{1}{6}} \frac{3dx}{\sqrt{1-9x^2}}$ **3**

(e) Find the obtuse angle, to the nearest minute, between the lines: **3**

$$4x - y + 6 = 0 \quad \text{and} \quad x + 3y - 7 = 0$$

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) If the equation $5x^3 - 6x^2 - 29x + 6 = 0$ has roots α, β and γ, find the value of: **2**

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

(b) Consider the functions:

$$f(x) = x\ln x - x \qquad x > 0$$
$$g(x) = 3 - x$$

(i) Find the stationary point of $y = f(x)$ and determine its nature. **2**

(ii) Draw the graph of $y = f(x)$. **2**

(iii) On the same graph of $y = f(x)$, draw the graph of $y = g(x)$ and hence explain why the equation $x\ln x - 3 = 0$ has only one root. **2**

(iv) Use one application of Newton's Method, with $x_1 = 2.8$, to find a better approximation of the root of the equation $x\ln x - 3 = 0$. **2**

(c) Gemma and Evan are in a group of nine people. **2**

How many groups of five may be selected so as to include one of Gemma or Evan but not both?

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{dx}{\sqrt{x}\,\sqrt{1+\sqrt{x}}}$ using the substitution $u = 1 + \sqrt{x}$. **3**

(b) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal to the parabola at the point P is **2**

$$x + py = 2ap + ap^3$$

(ii) If the normal at P cuts the y-axis at Q show that the co-ordinates of Q are $(0, 2a + ap^2)$. **1**

(iii) Show that the co-ordinates of R which divide the interval PQ externally in the ratio $2:1$ are $(-2ap, 4a + ap^2)$. **1**

(iv) Find the Cartesian equation of the locus of R and describe it geometrically. **3**

(v) Show that if the normal at P passes through a given point (h, k) then p must be a root of the equation: **1**

$$ap^3 + (2a - k)\,p - h = 0$$

(vi) Hence state the maximum number of normals to the parabola $x^2 = 4ay$ which can pass through any given point. **1**

Marks

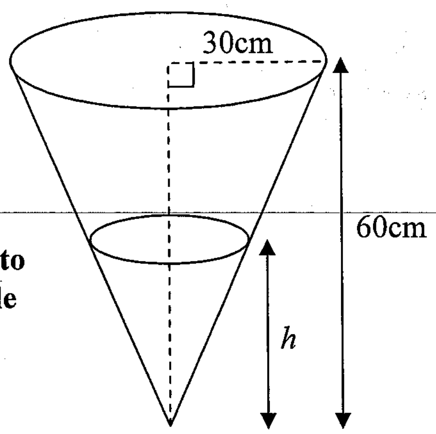
Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ where α is in radians. **2**

(ii) Hence, or otherwise, find the general solution of the equation **2**

$$\sqrt{3}\cos\theta - \sin\theta = 1$$

(b)



Not to scale

Water is being poured into a conical vessel at a constant rate of 36mLs^{-1}.

The radius of the vessel is 30cm and its height is 60cm.

After t seconds the depth of the water in the vessel is h cm.

(i) Show the volume of water in the vessel for any given h is given by: **2**

$$V = \frac{\pi h^3}{12}$$

(ii) What is the depth of water in the vessel after 4 seconds. (Give your answer to 3 significant figures.) **2**

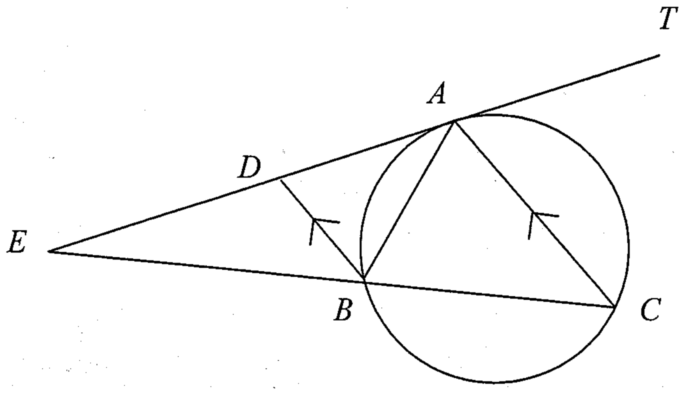
(iii) Find the rate at which the depth of water is increasing after 4 seconds. (Give your answer to 3 significant figures.) **2**

(iv) Find the rate at which the surface area S, **of the top of the water,** is changing when the depth is 32cm. **2**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)



ABC is a triangle inscribed in a circle. The tangent at A to the circle meets the side CB produced at E. The parallel from B to CA meets the tangent TE at D.

(i) Prove that ΔABE is similar to ΔEBD. **3**

(ii) Hence, or otherwise, show that $BE^2 = AE \times DE$. **2**

(b) The acceleration of a particle moving in a straight line is given by $\dfrac{d^2x}{dt^2} = -\dfrac{72}{x^2}$, where x metres is the displacement from the origin after t seconds.

Initially the particle is 9 metres to the right of the origin with a velocity of 4 metres per second.

(i) Show that the velocity V of the particle in terms of x is $V = \dfrac{12}{\sqrt{x}}$. **4**

Explain why V is always positive for the given initial conditions.

(ii) Find an expression for t in terms of x. **2**

(iii) How many seconds does it take for the particle to reach a point 35m to the right of the origin? **1**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) In a class there are 6 girls and 9 boys. Their classroom has 4 rows of 7 seats neatly arranged. Each student occupies a chair. Find the number of seating arrangements possible if:

(i) students can sit anywhere. **1**

(ii) all the girls want to occupy the first row. **1**

(iii) 3 particular girls and 4 particular boys fill the back row seated alternatively. **1**

(b) A golf ball is hit an angle α, where $0° < \alpha < 90°$. The initial velocity of the ball is V ms^{-1}. (Assume acceleration due to gravity $g = 9.8\text{ms}^{-2}$.)

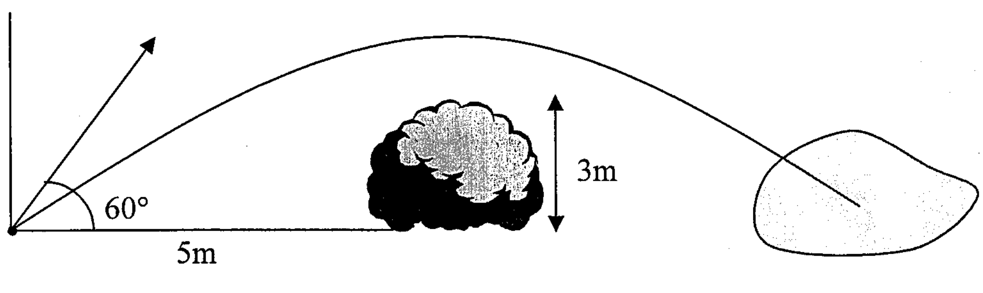
(i) Show that the horizontal and vertical displacement of the ball is given by: **2**

$$x = V\cos\alpha\, t$$
$$y = -\frac{1}{2}gt^2 + V\sin\alpha\, t$$

(ii) Show that $y = x\tan\alpha - \dfrac{gx^2}{2V^2}\left(1+\tan^2\alpha\right)$. **2**

To play one shot, Samuel must clear a 3m shrub which is 5m from his ball. He hits his ball so that it has an initial velocity of 10ms^{-1} and an angle of projection of $60°$.

Not to scale



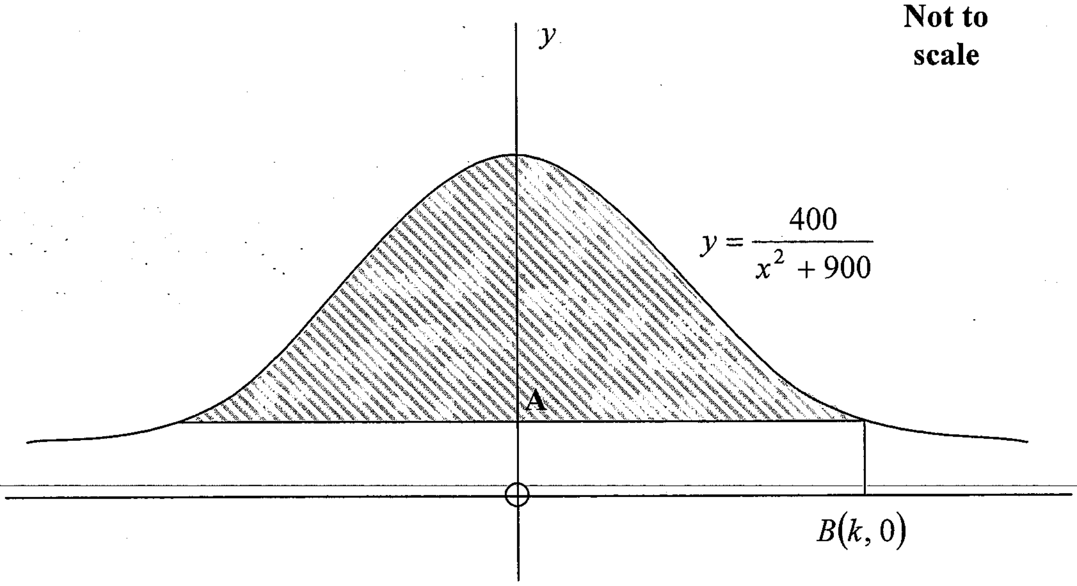
(iii) By how far does his ball clear the shrub? **2**

(iv) What is the horizontal distance travelled by the ball? **3**
(Assume the ball lands in a bunker of sand and stops immediately.)

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) The cross-section of a roof for a new-age, environmentally friendly building is described by the equation $y = \dfrac{400}{x^2 + 900}$.



(i) If A is the point $\left(0, \dfrac{1}{3}\right)$ find the value of k. **1**

(ii) Show that the shaded area is $\dfrac{20\left(2\pi - 3\sqrt{3}\right)}{9}$ m^2. **2**

(iii) By considering the integral $\displaystyle\int_{-k}^{k} \frac{400}{x^2 + 900}$ or otherwise, show that the area of the cross-section will never exceed $\dfrac{40\pi}{3}$ m^2. **2**

(b) (i) Show that $\sin x + \cos x = \sqrt{2} \sin\left(x + \dfrac{\pi}{4}\right)$. **2**

(ii) Prove that $\dfrac{d}{dx}\left(e^x \sin x\right) = \sqrt{2}\, e^x \sin\left(x + \dfrac{\pi}{4}\right)$. **2**

(iii) Prove by mathematical induction that if $y = e^x \sin x$, then $\dfrac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin\left(x + \dfrac{n\pi}{4}\right)$ where n is a positive integer. **3**

End of paper

2007 SCEGGS Trial HSC Examination - Mathematics Ext. 1 - Solutions

Q1 a) $\lim_{x\to 0} \frac{\sin 2x}{5x}$

$= \frac{1}{5} \lim_{x\to 0} \frac{\sin 2x}{x}$

$= \frac{2}{5} \lim_{x\to\infty} \frac{\sin 2x}{2x}$ ✓

Reas (2)

$= \frac{2}{5} \times 1$ ✓ as $\lim_{x\to\infty} \frac{\sin ax}{ax} = 1$

$= \frac{2}{5}$

b) $\frac{-2}{x-3} \le 1$

$(x-3)^2 \frac{-2}{x-3} \le 1(x-3)^2$

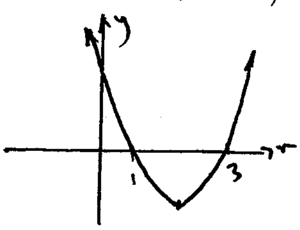
$-2(x-3) \le x^2-6x+9$

$-2x+6 \le x^2-6x+9$

$0 \le x^2-4x+3$ ✓

sketch $y = x^2-4x+3$

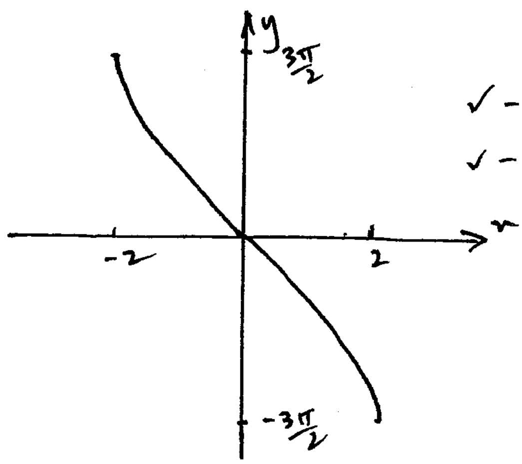
$= (x-3)(x-1)$



from graph $x \le 1$ or $x > 3$ ✓

c)

Comm (2)



✓ - for graph

✓ - for labels

Calc (3)

d) $\int_0^{\frac{1}{6}} \frac{3\,dx}{\sqrt{1-9x^2}} = \frac{3}{3}\int_0^{1/6} \frac{dx}{\sqrt{\frac{1}{9}-x^2}}$ ✓

$= \sin^{-1} 3x \Big]_0^{1/6}$ ✓

$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$ ✓

e) $4x - y + 6 = 0$ $\quad\quad$ $x + 3y - 7 = 0$

$y = 4x + 6$ $\quad\quad$ $3y = -x + 7$

$\therefore m = 4$ $\quad\quad$ $y = -\frac{1}{3}x + \frac{7}{3}$

$\therefore m = -\frac{1}{3}$ ✓

$\therefore \tan\theta = \left| \frac{4 - -\frac{1}{3}}{1 + 4 \times -\frac{1}{3}} \right|$ ✓

$= \left| \frac{13/3}{1 - 4/3} \right|$

$= |-13|$

$\theta = 85°36'$

$\therefore$ obtuse angle $= 180° - 85°36'$

$= 94°24'$ ✓

Be careful of the word obtuse!!

Q2 a) $5x^3 - 6x^2 - 29x + 6 = 0$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ ✓

$= \frac{c/a}{-d/a}$

$= \frac{c}{-d}$

$= \frac{-29}{-6}$ ✓

$= 4\,5/6$

- done fairly well
- the most common mistake was stating that $\frac{c}{a} = \frac{29}{5}$ $\therefore$ getting $-\frac{29}{6}$ as final answer

b) i) $y = x\ln x - x$

$$\frac{dy}{dx} = x.\frac{1}{x} + \ln x.1 - 1$$
$$= 1 + \ln x - 1$$
$$= \ln x$$

$\therefore$ st. points occur when $\frac{dy}{dx} = 0$

Calc (2)

$\therefore \ln x = 0$

$x = 1$

$y = -1$

$\therefore$ stationary point $(1, -1)$ ✓

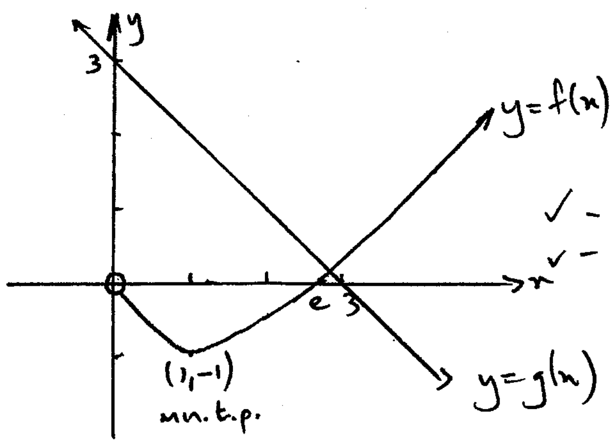
$$\frac{d^2y}{dx^2} = \frac{1}{x}$$

- Most candidates found the stationary point well but used the first derivative test to test nature. This is fine but it is time consuming and the 2nd derivative is easy to find.

when $x = 1$ $\frac{d^2y}{dx^2} = 1 > 0$ $\therefore (1,-1)$ is a minimum t.p. ✓

ii)

Comm (2)



✓ - for graph
✓ - for labels

- graph was done poorly students need to review sketching hard log graphs.

iii) The point of intersection represents the solution to $y = x\ln x - x$ and $y = 3 - x$ ✓

Comm (2)

$\therefore$ solving simultaneously

$$x\ln x - x = 3 - x$$
$$x\ln x - 3 = 0$$

since there is only one point of intersection then $x\ln x - 3 = 0$ has only one root. ✓

- poorly explained students should draw the conclusion between solving simultaneously and one point of intersection.

iv) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\quad f(x) = x\ln x - 3$

$f'(x) = 1 + \ln x$ ✓

$= 2\cdot8 - \frac{2\cdot8\ln(2\cdot8) - 3}{1 + \ln 2\cdot8}$

$= 2\cdot86$ (to 2 dec. pl.) ✓

- Many silly errors made here!

eg $f(x) = x\ln x - x$

or

$f'(x) = \ln x$.

Reas (2)

c) No. of groups of 5 $= {}^7C_4 \times 2$ ✓

$= 35 \times 2$

$= 70$ ✓

- done poorly

students must review this work as there will be at least one question in the HSC

Q3 a) $\int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}}$ $\quad u = 1 + x^{1/2}$

$du = \frac{1}{2}x^{-1/2}\,dx$

$= \frac{1}{2\sqrt{x}}\,dx$ ✓

$= 2\int \frac{dx}{2\sqrt{x}\sqrt{1+\sqrt{x}}}$

$= 2\int \frac{du}{\sqrt{u}}$

Calc (3)

$= 2\int u^{-1/2}\,du$ ✓

$= 2 \times \frac{u^{1/2}}{1/2} + c$

$= 4\sqrt{u} + c$

$= 4\sqrt{1+\sqrt{x}} + c$ ✓

Remember to write in terms of $\underline{\underline{x}}$

b) i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

when $x = 2ap$

$\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$

$\therefore M_{tang} = p$ ✓

$\therefore M_{norm} = \frac{-1}{p}$

$\therefore$ using $y - y_1 = m(x - x_1)$

(comm (2))

$y - ap^2 = -\frac{1}{p}(x - 2ap)$ ✓

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$

ii) $Q: x = 0$ sub into $x + py = 2ap + ap^3$

$\therefore 0 + py = 2ap + ap^3$ ✓

$y = 2a + ap^2$

$\therefore Q(0, 2a + ap^2)$

iii) $P(2ap, ap^2)$ $Q(0, 2a + ap^2)$

$2 : -1$

$R\left(\frac{2 \times 0 + -1 \times 2ap}{2 + -1}, \frac{2(2a + ap^2) - 1 \times ap^2}{2 + -1}\right)$ ✓

$R\left(\frac{-2ap}{1}, \frac{4a + 2ap^2 - ap^2}{1}\right)$

$R(-2ap, 4a + ap^2)$

iv) $X = -2ap \ldots ①$ $Y = 4a + ap^2 \ldots ②$

$p = \frac{X}{-2a} \ldots ③$

sub ③ into ②

(comm (3))

$Y = 4a + a\left(\frac{X}{-2a}\right)^2$ ✓

$= 4a + \frac{aX^2}{4a^2}$

$= 4a + \frac{X^2}{4a}$

$4aY = 16a^2 + X^2$

$X^2 = 4aY - 16a^2$

$X^2 = 4a(Y - 4a)$ ✓

the locus of R is a parabola with a vertex at $(0, 4a)$ ✓

Must state more than the fact that it is a parabola

v) (h,k) must satisfy $x+py = 2ap+ap^3$

$$\therefore\ h+pk = 2ap+ap^3 \quad \checkmark$$
$$ap^3+2ap-pk-h=0$$
$$ap^3+(2a-k)p-h=0$$

Comm(1) vi) since this equation is a cubic the maximum number of solution it can have is 3. ✓

$\therefore$ the maximum number of normals which can pass through a point is 3.

Q4 i) $\sqrt{3}\cos\theta - \sin\theta = R\cos(\theta+\alpha)$

$$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$\therefore\ R\cos\alpha = \sqrt{3}\ldots(1)$

$R\sin\alpha = 1\ldots(2)$

$(2)\div(1)\quad \tan\alpha = \frac{1}{\sqrt{3}}$ ✓

Comm(2)

$\alpha = \frac{\pi}{6}$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3+1$

$R^2 = 4$

$R = 2$ ✓

$\therefore\ \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta+\frac{\pi}{6}\right)$

- done well by most candidates

ii) $\sqrt{3}\cos\theta - \sin\theta = 1$

$\therefore\ 2\cos\left(\theta+\frac{\pi}{6}\right) = 1$

$\cos\left(\theta+\frac{\pi}{6}\right) = \frac{1}{2}$

$\theta + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

$\therefore\ \theta = 2n\pi + \frac{\pi}{6}$ ✓

or $\theta = 2n\pi - \frac{\pi}{2}$ ✓

- done poorly students must learn general solution results

b) let r = the radius of the cone of water

$\therefore$ by similar triangles

$$\frac{r}{h} = \frac{30}{60} = \frac{1}{2} \quad ✓$$

$$r = \frac{h}{2}$$

Reas (2)

$$\therefore V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 . h \quad ✓$$

$$= \frac{1}{3}\pi\frac{h^3}{4} = \frac{\pi h^3}{12}$$

- many students didn't see the relationship between r and h.

ii) $\frac{dV}{dt} = 36\ cm^3 s^{-1}$

$$V = 36 \times 4 = 144 cm^3 \quad ✓$$

Reas (2)

$$\therefore 144 = \frac{\pi h^3}{12}$$

$$1728 = \pi h^3$$

$$h^3 = \frac{1728}{\pi}$$

$$h = 8.19 cm \quad ✓$$

- ~~some students made this~~ question harder than it needed to be. You don't need to integrate to find the volume.

iii) $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ $\qquad V = \frac{\pi h^3}{12}$, $\frac{dV}{dh} = \frac{\pi h^2}{4}$ ✓

Calc (2)

$$= \frac{4}{\pi h^2} \times 36$$

$$= \frac{4}{\pi \times 8.19^2} \times 36$$

$$= 0.683 cms^{-1} \quad ✓$$

- Most students need to revise rates of change with two variables.

iv) $S = \pi r^2$

$$\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$$

$$S = \pi\left(\frac{h}{2}\right)^2 = \frac{\pi h^2}{4}$$

$$\frac{dS}{dh} = \frac{\pi h}{2} \quad ✓$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \times 36$$

Calc (2)

$$= \frac{4}{\pi \times 32^2} \times 36$$
$$= 0.04476\ldots$$

$$\frac{ds}{dt} = \frac{\pi h}{2} \times 0.04976\ldots$$
$$= \frac{\pi \times 32}{2} \times 0.04476\ldots$$
$$= 2.25 \text{ cm}^2\text{s}^{-1}$$ ✓

Q5 a) i) $\angle EAB = \angle ACB$ (angle between tang. and chord at pt. of contact eq. angle in alt. segment) ✓

Reas (3)

$\angle ACB = \angle DBE$ (corr. angles in parallel lines are equal) ✓

$\therefore \angle EAB = \angle DBE$

$\angle E$ is common

$\therefore \triangle ABE \ ||| \ \triangle BDE$ (equiangular) ✓

ii) $\frac{AE}{BE} = \frac{BE}{DE}$ ✓ (corr. sides in similar triangles are in the same ratio) ✓

Reas (2)

$BE^2 = AE \times DE.$

b) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{72}{x^2}$

$$\frac{1}{2}v^2 = \int -72x^{-2}\,dx$$
$$= 72x^{-1} + c$$ ✓

when $x = 9$ $v = 4$

$$\frac{1}{2}4^2 = \frac{72}{9} + c$$
$$8 = 8 + c$$
$$\therefore c = 0$$ ✓

Remember 'c'!! use conditions to find the value

Calc (3)

$$\frac{1}{2}v^2 = \frac{72}{x}$$

$$v^2 = \frac{144}{x}$$

$$v = \pm\frac{12}{\sqrt{x}}$$ ✓

the particle starts at $x = 9$ and is travelling at $4\,ms^{-1}$ to the right

Comm (1)

since $\frac{12}{\sqrt{x}} > 0$ for all x as x is positive initially then v will remain positive for all x as the particle will not stop. ✓

must explain that the particle cannot stop

$$\therefore v = \frac{12}{\sqrt{x}}$$

ii) $$\frac{dx}{dt} = \frac{12}{\sqrt{x}}$$

$$\frac{dt}{dx} = \frac{x^{1/2}}{12}$$

$$t = \int \frac{x^{1/2}}{12}\,dx$$

Calc (2)

$$= \frac{x^{3/2}}{18} + c$$ ✓

remember c

when $t = 0$ $x = 9$

$$0 = \frac{9^{3/2}}{18} + c$$

$$= \frac{3}{2} + c$$

$$c = -\frac{3}{2}$$

$$\therefore t = \frac{\sqrt{x^3}}{18} - \frac{3}{2}$$ ✓

iii) when $x = 35$

Calc (1)

$$t = \frac{\sqrt{35^2}}{18} - \frac{3}{2}$$

$= 10.0$ (to 1 d.p.)

$\therefore$ it takes 10 seconds ✓

Q6 a) i) ${}^{28}P_{15}$ ✓

Recs (3)

ii) ${}^{7}P_{6} \times {}^{21}P_{9}$ ✓

iii) ${}^{4}P_{4} \times {}^{3}P_{3} \times {}^{21}P_{8}$ ✓

- every student must review this work!!!

b) i) horz

$\ddot{x} = 0$

$\dot{x} = c_1$

$t=0 \quad \dot{x} = V\cos\alpha \quad \therefore c_1 = V\cos\alpha$

$\therefore \dot{x} = V\cos\alpha$

$x = V\cos\alpha\, t + c_2$ ✓

$\cancel{t=0 \quad x=0 \quad \therefore c_2=0}$

$x = V\cos\alpha\, t \quad \ldots (1)$

vert

$\ddot{y} = -g$

$\dot{y} = -gt + k_1$

$t=0 \quad \dot{y} = V\sin\alpha \quad \therefore k_1 = V\sin\alpha$

$\dot{y} = -gt + V\sin\alpha$

$y = \frac{-gt^2}{2} + V\sin\alpha\, t + k_2$ ✓

$t=0 \quad y=0 \quad \therefore k_2 = 0$

$\therefore y = \frac{-gt^2}{2} + V\sin\alpha\, t \ldots (2)$

Calc (2)

- done very well. be careful not to leave steps out when rushing to answer questions

ii) from (1) $t = \frac{x}{V\cos\alpha}$

sub into (2)

$$y = \frac{-g}{2}\left(\frac{x}{V\cos\alpha}\right)^2 + V\sin\alpha \cdot \frac{x}{V\cos\alpha} \quad ✓$$

$$= \frac{-g}{2}\frac{x^2}{V^2\cos^2\alpha} + x\tan\alpha$$

$$= \frac{-gx^2}{2V^2}\sec^2\alpha + x\tan\alpha \quad ✓$$

$$\therefore y = x\tan\alpha - \frac{gx^2}{2V^2}(1+\tan^2\alpha)$$

Calc (2)

- done very well

iii) find y when $\alpha = 60°$ $V=10$ $x=5$

$$y = 5\tan 60° - \frac{9.8\times 5^2}{2\times 10^2}\times(1+\tan^2 60) \quad ✓$$

$$= 3.76\text{ m} \quad (\text{to 2 d.p.})$$

$\therefore$ it clears the shrub by approx 76cm. ✓

Calc (2)

- many students didn't use ii) for this question - that's fine - just took a little longer

ii) find t when $y=0$

$0 = -\frac{1}{2} \times 9.8 \times t^2 + 10 \sin 60° t$ ✓

$= t(-4.9t + 5\sqrt{3})$

Calc (3)

$\therefore t = 0$ (initial) $\quad t = \frac{5\sqrt{3}}{4.9}$ ✓

$\therefore t = 1.767$ (to 4 sig figs)

$x = 10 \cos 60° \times 1.767$

$= 8.84$ m (to 2 d.p.) ✓

- done fairly well

Q7 a) i) find x when:

$$\frac{1}{3} = \frac{400}{x^2+900}$$

$$x^2+900 = 1200$$

$$x^2 = 300$$

$$x = \pm 10\sqrt{3}$$

since B is located above the positive x-axis

$x = 10\sqrt{3}$ ✓

ii) Area of shaded region =

Area under curve − area of rectangle

Area under curve $= 2\int_0^{10\sqrt{3}} \frac{400\,dx}{x^2+900}$ (even fn.)

$$= 800\int_0^{10\sqrt{3}} \frac{dx}{x^2+900}$$

$$= \left[\frac{800}{30}\tan^{-1}\frac{x}{30}\right]_0^{10\sqrt{3}}$$ ✓

Remember the rectangle as well

$$= \frac{80}{3}\left(\tan^{-1}\frac{10\sqrt{3}}{30} - \tan^{-1}0\right)$$

$$= \frac{80}{3}\left(\frac{\pi}{6} - 0\right)$$

$$= \frac{40\pi}{9}$$

Area of rectangle $= 20\sqrt{3} \times \frac{1}{3}$

$= \frac{20\sqrt{3}}{3}$

Calc(2) $\therefore$ Area $= \frac{40\pi}{9} - \frac{20\sqrt{3}}{3}$ ✓

$= \frac{20(2\pi - 3\sqrt{3})}{9} \text{ m}^2$

iii) Area $= 2\int_0^k \frac{400}{x^2+900} - 2k \times \frac{400}{k^2+900}$

$= \frac{80}{3}\left[\tan^{-1}\frac{x}{30}\right]_0^k - \frac{800k}{k^2+900}$

Must discuss the limit of the rectangle areas.

$= \frac{80}{3}\tan^{-1}\frac{k}{30} - \frac{800k}{k^2+900}$

Comm(2) as $k \to \infty$ $\tan^{-1}\frac{k}{30} \to \frac{\pi}{2}$ & $\frac{800k}{k^2+900} \to 0$ ✓

$\therefore$ limiting area $= \frac{80}{3} \times \frac{\pi}{2}$ ✓

$= \frac{40\pi}{3} \text{ m}^2$

b) i) $\sin x + \cos x = R\sin(x+\alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$

$\therefore$ $R\sin\alpha = 1 \ldots①$

$R\cos\alpha = 1 \ldots②$

① ÷ ② $\tan\alpha = 1$ ✓

$\alpha = \frac{\pi}{4}$

$R^2\sin^2\alpha + R^2\cos^2\alpha = 1^2 + 1^2$ ✓

$R^2 = 2$

$R = \sqrt{2}$

$\therefore$ $\sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$

ii) $\frac{d}{dx}(e^x \sin x) = e^x\cos x + e^x \sin x$ ✓

$= e^x(\sin x + \cos x)$

$= e^x . \sqrt{2}\sin(x + \frac{\pi}{4})$ ✓

$= \sqrt{2}e^x\sin(x+\frac{\pi}{4})$

iii) Step 1: Show the result is true for $n=1$

i.e. $\frac{dy}{dx} = (\sqrt{2})^1 e^x \sin\left(x + \frac{1\pi}{4}\right)$

$y = e^x \sin x$

$= \sqrt{2} e^x \sin\left(x + \frac{\pi}{4}\right)$ from ii)

$\therefore$ result is true for $n=1$

Step 2: Assume the result is true for $n=k$

Reas (3).

i.e. $\frac{d^k y}{dx^k} = (\sqrt{2})^k e^x \sin\left(x + \frac{k\pi}{4}\right)$

Step 3: Show the result is true for $n=k+1$

i.e. $\frac{d^{k+1} y}{dx^{k+1}} = (\sqrt{2})^{k+1} e^x \sin\left(x + \frac{(k+1)\pi}{4}\right)$

$$\text{LHS} = \frac{d^{k+1} y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left((\sqrt{2})^k e^x \sin\left(x + \frac{k\pi}{4}\right)\right) \checkmark$$

$$= (\sqrt{2})^k \left[e^x \sin\left(x + \frac{k\pi}{4}\right) + e^x \cos\left(x + \frac{k\pi}{4}\right)\right] \checkmark$$

$$= (\sqrt{2})^k \left[e^x (\sqrt{2})^x \sin\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right)\right]$$

from i)

$$= (\sqrt{2})^{k+1} e^x \sin\left(x + \frac{(k+1)\pi}{4}\right) \checkmark$$

$= \text{RHS}$

$\therefore$ result is true for $n=k+1$

$\therefore$ By the principle of mathematical induction the result is true for all positive integers n.